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Twistor spinors on Lorentzian symmetric spaces

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Abstract

An indecomposable Riemannian symmetric space which admits non-trivial twistor spinors has constant sectional curvature. Furthermore, each homogeneous Riemannian manifold with parallel spinors is flat. In the present paper we solve the twistor equation on all indecomposable Lorentzian symmetric spaces explicitly. In particular, we show that there are — in contrast to the Riemannian case — indecomposable Lorentzian symmetric spaces with twistor spinors, which have non-constant sectional curvature and non-flat and non-Ricci flat homogeneous Lorentzian manifolds with parallel spinors. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let (M^n, g) be an oriented semi-Riemannian spin manifold with the spinor bundle S. The twistor operator $\mathcal{D}$ is defined as the composition of the spinor derivative ∇^S with the projection p onto the kernel of the Clifford multiplication μ:

$$\mathcal{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{=} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu).$$

The solutions of the conformally invariant equation $\mathcal{D}\varphi = 0$ are called twistor spinors. Twistor spinors were introduced by Penrose in General Relativity (see [12–14]). They are related to Killing vector fields in semi-Riemannian supergeometry (see [1]). In the last few years essential results concerning the geometry of Riemannian spin manifolds admitting twistor spinors were obtained by A. Lichnerowicz, T. Friedrich, K. Habermann, H.-B. Rademacher, W. Kuehnel and other authors. For a survey on the literature cf. [3,10]. In the

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Lorentzian setting a relation between a certain class of solutions of the twistor equation and the Fefferman spaces occurring in CR-geometry was established (cf. [2,11]). Special solutions of the Lorentzian twistor equation, the Killing spinors, were studied by Bohle [4]. Killing spinors in the general pseudo-Riemannian setting were discussed by Kath [9].

In the present paper we study the twistor equation on Lorentzian symmetric spaces. Using the explicit classification results we determine the twistor spinors on all indecomposable Lorentzian symmetric spaces explicitly. Let us remark that an indecomposable (=irreducible) *Riemannian* symmetric space which admits non-trivial twistor spinors has constant sectional curvature. We will see below that the same is true for *irreducible* Lorentzian symmetric spaces. But in the Lorentzian signature, there is second type of indecomposable symmetric spaces, which are non-irreducible and have solvable transvection group. This type of Lorentzian symmetric spaces admits solutions of the twistor equation although it has non-constant sectional curvature. Furthermore, we will see that there are non-flat and non-Ricci-flat Lorentzian symmetric spaces with parallel spinors. In contrast to that, each homogeneous *Riemannian* manifold with parallel spinors has to be flat.

Let $\mathcal{T}(M^n, g)$ denote the space of all twistor spinors of (M^n, g) and let $n \geq 3$. It is known that

$$\dim \mathcal{T}(M^n, g) \leq 2 \cdot 2^{[n/2]}$$

(see [3]). If (M^n, g) is conformally flat and simply connected, then one has $\dim \mathcal{T}(M^n, g) = 2 \cdot 2^{[n/2]}$. In the present paper we prove, in particular

1. If (M^n, g) is an indecomposable non-conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$, then each twistor spinor is parallel and $\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]}$, where $q = \frac{1}{2}, \frac{1}{4}$ or 0, depending on the fundamental group $\pi_1(M)$ and on the spin structure of (M^n, g).
2. If (M^n, g) is an indecomposable conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$ and non-constant sectional curvature, then $\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]}$, where $q = 2, \frac{3}{2}, 1, \frac{3}{4}$ or 0, depending on $\pi_1(M)$ and on the spin structure.
3. If (M^n, g) is a Lorentzian symmetric spin manifold of dimension $n \geq 3$ and constant sectional curvature, then $\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]}$, where $q = 2, 1$, or 0, depending on $\pi_1(M)$ and on the spin structure.

2. Lorentzian symmetric spaces

Let us first recall the description of Lorentzian symmetric spaces. A connected semi- Riemannian manifold (M^n, g) is called *indecomposable*, if there is no proper, non-degenerate subspace of $T_x M$ invariant under the action of the holonomy group $\mathrm{Hol}_x(g)$. Each simply connected semi-Riemannian symmetric space is isometric to a product $M_0 \times M_1 \times \cdots \times M_r$, where M_i, $i = 1, \ldots, r$, are indecomposable simply connected semi-Riemannian symmetric spaces of dimension ≥ 2 and M_0 is semi-Euclidean.

Let (M^n, g) be a Lorentzian symmetric space. By $G(M)$ we denote the group of transvections of (M^n, g) and by $\mathfrak{g}$ its Lie algebra. One has the following structure result.

Theorem 1 ([8]). *Let (M^n, g) be an indecomposable Lorentzian symmetric space of dimension $n \geq 2$. Then the Lie algebra $\mathfrak{g}$ of the transvection group of (M^n, g) is either semi-simple or solvable.*

Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-2})$ be an $(n-2)$-tupel of real numbers $\lambda_j \in \mathbb{R}\backslash\{0\}$ and let us denote by $M^n_{\underline{\lambda}}$ the Lorentzian space $M^n_{\underline{\lambda}} := (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} := 2\,\mathrm{d}s\,\mathrm{d}t + \sum_{j=1}^{n-2} \lambda_j x_j^2\,\mathrm{d}s^2 + \sum_{j=1}^{n-2} \mathrm{d}x_j^2.$$

If $\underline{\lambda}_\pi = (\lambda_{\pi(1)}, \dots, \lambda_{\pi(n-2)})$ is a permutation of $\underline{\lambda}$ and $c > 0$, then $M^n_{\underline{\lambda}}$ is isometric to $M^n_{c\underline{\lambda}_\pi}$.

Theorem 2 ([7,8]). *Let (M^n, g) be an indecomposable solvable Lorentzian symmetric space of dimension $n \geq 3$. Then (M^n, g) is isometric to $M^n_{\underline{\lambda}}/A$, where $\underline{\lambda} \in (\mathbb{R}\backslash\{0\})^{n-2}$ and A is a discrete subgroup of the centralizer $Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$ of the transvection group $G(M_{\underline{\lambda}})$ in the isometry group $I(M_{\underline{\lambda}})$ of $M^n_{\underline{\lambda}}$.*

For the centralizer $Z_{\underline{\lambda}} := Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$ the following theorem is known.

Theorem 3 ([5]). *Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-2})$ be a tupel of non-zero real numbers.*
1. *If there is a positive λ_i or if there are two numbers λ_i, λ_j such that $\lambda_i/\lambda_j \notin \mathbb{Q}^2$, then $Z_{\underline{\lambda}} \simeq \mathbb{R}$ and $\varphi \in Z_{\underline{\lambda}}$ if and only if $\varphi(s, t, x) = (s, t + \alpha, x), \alpha \in \mathbb{R}$.*
2. *Let $\lambda_i = -k_i^2 < 0$ and $k_i/k_j \in \mathbb{Q}$ for all $i, j \in \{1, \dots, n-2\}$. Then $\varphi \in Z_{\underline{\lambda}}$ if and only if*

$$\varphi(s, t, x) = \left(s + \beta, t + \alpha, (-1)^{m_1} x_1, \dots, (-1)^{m_{n-2}} x_{n-2}\right),$$

where $\alpha \in \mathbb{R}, m_1, \dots, m_{n-2} \in \mathbb{Z}$ and $\beta = m_i \pi/k_i$ for all $i = 1, \dots, n-2$.

Let us denote by $S^n_1(r)$ the pseudo-sphere

$$S^n_1(r) := \{x \in \mathbb{R}^{n+1,1} | \langle x, x\rangle_{n+1,1} = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2\} \subset \mathbb{R}^{n+1,1},$$

and by $H^n_1(r)$ the pseudo-hyperbolic space

$$H^n_1(r) := \{x \in \mathbb{R}^{n+1,2} | \langle x, x\rangle_{n+1,2} = -x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = -r^2\} \subset \mathbb{R}^{n+1,2},$$

with the Lorentzian metrics induced by $\langle\cdot,\cdot\rangle_{n+1,1}$ and $\langle\cdot,\cdot\rangle_{n+1,2}$, respectively.

Theorem 4 ([6,15]). *Let (M^n, g) be an indecomposable semi-simple Lorentzian symmetric space of dimension $n \geq 3$. Then (M^n, g) has constant sectional curvature $k \neq 0$. Therefore, it is isometric to $S^n_1(r)/_{\{\pm I\}}$ or $S^n_1(r)$ $(k = 1/r^2 > 0)$, or to a Lorentzian covering of $H^n_1(r)/_{\{\pm I\}}$ $(k = -1/r^2 < 0)$.*

3. Spinor representation

For concrete calculations we will use the following realization of the spinor representation. Let $\mathrm{Cliff}_{n,k}$ be the Clifford algebra of $(\mathbb{R}^n, -\langle\cdot,\cdot\rangle_k)$, where $\langle\cdot,\cdot\rangle_k$ is the scalar product $\langle x, y\rangle_k := -x_1y_1 - \cdots - x_ky_k + x_{k+1}y_{k+1} + \cdots + x_ny_n$. For the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ one has the following relations in $\mathrm{Cliff}_{n,k}$: $e_i \cdot e_j + e_j \cdot e_i = -2\varepsilon_j\delta_{ij}$, where $\varepsilon_j = -1$ when $j \le k$ and $\varepsilon_j = 1$ when $j > k$. Denote

$$\tau_j = \begin{cases} \mathrm{i}, & j \le k, \\ 1, & j > k, \end{cases}$$

and

$$U = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

If $n = 2m$ is even, we have an isomorphism

$$\phi_{2m,k} : \mathrm{Cliff}^{\mathbb{C}}_{2m,k} \xrightarrow{\sim} M(2^m; \mathbb{C})$$

given by the Kronecker product

$$\begin{aligned} \phi_{2m,k}(e_{2j-1}) &= \tau_{2j-1} \quad E \otimes \cdots \otimes E \otimes U \otimes T \otimes \cdots \otimes T, \\ \phi_{2m,k}(e_{2j}) &= \tau_{2j} \quad\quad E \otimes \cdots \otimes E \otimes V \otimes \underbrace{T \otimes \cdots \otimes T}_{j-1}. \end{aligned} \tag{1}$$

If $n = 2m + 1$ is odd, we have the isomorphism

$$\phi_{2m+1,k} : \mathrm{Cliff}^{\mathbb{C}}_{2m+1,k} \xrightarrow{\sim} M(2^m; \mathbb{C}) \oplus M(2^m; \mathbb{C})$$

given by

$$\begin{aligned} \Phi_{2m+1,k}(e_k) &= (\Phi_{2m,k}(e_k), \Phi_{2m,k}(e_k)), \quad k = 1, \ldots, 2m, \\ \Phi_{2m+1,k}(e_n) &= \tau_n(\mathrm{i}T \otimes \cdots \otimes T, -\mathrm{i}T \otimes \cdots \otimes T). \end{aligned} \tag{2}$$

Let $\mathrm{Spin}(n, k) \subset \mathrm{Cliff}_{n,k}$ be the spin group. The spinor representation is given by

$$k_{n,k} = \hat{\phi}_{n,k}|_{\mathrm{Spin}_{(n,k)}} : \mathrm{Spin}(n, k) \to \mathrm{GL}(\mathbb{C}^{2^{[n/2]}}).$$

where $\hat{\phi}_{n,k} = \phi_{n,k}$ when $n = 2m$ and $\hat{\phi}_{n,k} = \mathrm{proj}_1 \circ \phi_{n,k}$ when $n = 2m + 1$. We denote this representation by $\Delta_{n,k}$. If $n = 2m$, $\Delta_{2m,k}$ splits into the sum $\Delta_{2m,k} = \Delta^+_{2m,k} \oplus \Delta^-_{2m,k}$, where $\Delta^{\pm}_{2m,k}$ are the eigenspaces of the endomorphism $\phi_{2m,k}(e_1, \ldots, e_{2m})$ to the eigenvalue $\pm\mathrm{i}^{m+k}$. Let us denote by $u(\delta) \in \mathbb{C}^2$ the vector $u(\delta) = (1/\sqrt{2})\binom{1}{-\delta\mathrm{i}}$, $\delta = \pm 1$, and let

$$u(\delta_1, \ldots, \delta_m) = u(\delta_1) \otimes \cdots \otimes u(\delta_m), \quad \delta_j = \pm 1. \tag{3}$$

Then $(u(\delta_1, \ldots, \delta_m) | \prod_{j=1}^m \delta_j = \pm 1)$ is an orthonormal basis of $\Delta^{\pm}_{2m,k}$ with respect to the standard scalar product of $\mathbb{C}^{2^m}$.

4. General properties of twistor spinors

In this section we recall some properties of twistor spinors which we will need in the following calculations. For proofs see [3]. Let (M^n, g) be an oriented semi-Riemannian spin manifold of dimension $n \geq 3$. We denote by S the spinor bundle of (M^n, g), by $\nabla^s : \Gamma(S) \to \Gamma(T^*M \otimes S)$ the spinor derivative given by the Levi-Civita connection of (M^n, g) and by $D : \Gamma(S) \to \Gamma(S)$ the Dirac operator on S. Let $p : TM \otimes S \to TM \otimes S$ be the projection onto the kernel of the Clifford multiplication μ. The *twistor operator*

$$\mathcal{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \stackrel{g}{=} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu)$$

is locally given by

$$\mathcal{D}\varphi = \sum_{k=1}^{n} \varepsilon_k s_k \otimes \left(\nabla^S_{s_k} + \frac{1}{n} s_k \cdot D\varphi \right),$$

where $(s_1, \ldots, s_n)$ is a local orthonormal basis and $\varepsilon_k = g(s_k, s_k) = \pm 1$. A spinor field $\varphi \in \Gamma(S)$ is called twistor spinor if $\mathcal{D}\varphi = 0$.

Proposition 1. *For a spinor field $\varphi \in \Gamma(S)$, the following conditions are equivalent:*

1. *φ is a twistor spinor.*
2. *φ satisfies the so-called twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \tag{4}$$

for all vector fields X.

3. *There exists a spinor field $\psi \in \Gamma(S)$ such that*

$$\psi = g(X, X) X \cdot \nabla^S_X \varphi \tag{5}$$

for all vector fields X with $|g(X, X)| = 1$.

The dimension of the space $\mathcal{T}(M^n, g)$ of all twistor spinors is conformally invariant and bounded by

$$\dim \mathcal{T}(M^n, g) \leq 2 \cdot 2^{[n/2]}.$$

If (M^n, g) is simply connected and conformally flat, $\dim \mathcal{T}(M^n, g) = 2 \cdot 2^{[n/2]}$. In particular, the twistor spinors on the semi-Euclidean space $(\mathbb{R}^{n,k}, \langle , \rangle_{n,k})$ are given by

$$\mathcal{T}(\mathbb{R}^{n,k}) = \{\varphi \in C^\infty(\mathbb{R}^{n,k}, \Delta_{n,k}) | \varphi(x) = u + x \cdot v; u, v \in \Delta_{n,k}\}.$$

Let R be the scalar curvature and Ric be the Ricci curvature of (M^n, g). $K : TM \to TM$ denotes the $(1, 1)$-Schouten tensor of (M^n, g)

$$K(X) = \frac{1}{n-2} \left(\frac{R}{2(n-1)} X - \mathrm{Ric}(X) \right).$$

Furthermore, let W be the $(4, 0)$-Weyl tensor of (M, g) and let us denote by the same symbol the corresponding $(2, 2)$-tensor field $W : \Lambda^2 M \to \Lambda^2 M$. Then we have the following proposition.

Proposition 2. *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then*

$$\nabla^S_X D\varphi = \frac{n}{2} K(X) \cdot \varphi, \tag{6}$$

$$W(\eta) \cdot \varphi = 0 \tag{7}$$

for all vector fields X and 2-forms η.

Finally, we recall two possibilities to obtain new manifolds with twistor spinors from a given one.

Let $(\tilde{M}^n, \tilde{g})$ be a simply connected parallelizable semi-Riemannian manifold and let $A \subset I(\tilde{M}, \tilde{g})$ denote a discrete subgroup of orientation preserving isometries of $(\tilde{M}, \tilde{g})$. We trivialize the spin structure of $(\tilde{M}, \tilde{g})$ with respect to a fixed global orthogonal basis field $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_n)$. For $\gamma \in A$ we denote by $\Gamma(x) \in SO(n, k)$ the matrix of $\mathrm{d}\gamma_x$ with respect to $\mathfrak{a}(x)$ and $\mathfrak{a}(\gamma(x))$. Then there are two lifts $\tilde{\Gamma}^{\pm}$ of Γ into Spin (n, k).

$$\begin{tikzcd} & Spin(n,k) \arrow[d, "\lambda"] \\ \tilde{M} \arrow[ur, "\tilde{\Gamma}^{\pm}"] \arrow[r, "\Gamma"'] & SO(n,k) \end{tikzcd}$$

Let $\mathcal{E}(A)$ be the set of all left actions of A on $\tilde{M} \times \mathrm{Spin}(n, k)$ such that

$$\varepsilon(\gamma)(x, a) = (\gamma(x), \varepsilon(\gamma, x) \cdot a) \quad \text{and} \quad \varepsilon(\gamma, x) = \tilde{\Gamma}(x)^{\pm}.$$

The set of left actions $\mathcal{E}(A)$ corresponds to the set of spinor structures of the oriented semi-Riemannian manifold $M = \tilde{M}/A$. The spinor bundle on M corresponding to $\varepsilon \in \mathcal{E}(A)$ is given by

$$S_\varepsilon = \tilde{M} \times \Delta_{n,k}|_{(A,\varepsilon)},$$

where $\varepsilon(\gamma)(x, v) = (\gamma(x), \varepsilon(\gamma, x) \cdot v)$ for all $\gamma \in A$. Hence, the spinor fields on M corresponding to $\varepsilon \in \mathcal{E}(A)$ are given by the ε-invariant functions

$$\Gamma(S_\varepsilon) = C^\infty(\tilde{M}, \Delta_{n,k})^\varepsilon := \{\varphi \in C^\infty(\tilde{M}, \Delta_{n,k}) | \varphi(\gamma(x)) = \varepsilon(\gamma, x) \cdot \varphi(x)\},$$

and for the twistor spinors on M with the spin structure ε we have the following proposition.

Proposition 3. *The twistor spinors on $M = \tilde{M}/A$ with respect to the spin structure $\varepsilon \in \mathcal{E}(A)$ are given by*

$$\mathcal{T}((M, g), \varepsilon) = \{\varphi \in \mathcal{T}(\tilde{M}, \tilde{g}) | \varphi \text{ is } \varepsilon\textit{-invariant}\}.$$

Let (M^{n+1}, g) be a semi-Riemannian spin manifold with spinor bundle S_M and let $F^n \subset M^{n+1}$ be a non-degenerate oriented hypersurface in M^{n+1}. We denote by $\eta : F \to TM$ the Gauss map of $F, \kappa(\eta) := g(\eta, \eta) = \pm 1$. It is well known that the spinor bundle S_F of $(F, g|_F)$ with respect to the spin structure on F induced by the embedding is isomorphic to $S_{M|F}$ in case of even dimension n and to $S_M^{\pm}|_F$ in case of odd dimension n. Using this identification the Clifford multiplication and the spinor derivative are expressed by

$$X \cdot (\varphi|_F) = (X \cdot \varphi)|_F, \qquad \nabla_X^{S_F}(\varphi|_F) = \left(\nabla_X^{S_M^{(\pm)}} \varphi + \frac{1}{2}\kappa(\eta)\nabla_X^M \eta \cdot \eta \cdot \varphi \right)\Big|_F,$$

where $\varphi \in \Gamma(S_M^{(\pm)}),\ X \in TF$ and $\varphi|_F$ always means the spinor field in $\Gamma(S_F)$ corresponding to φ with respect to the above-mentioned isomorphism.

Proposition 4. *If $F^n \subset M^{n+1}$ is an umbilic hypersurface and $\varphi \in \Gamma(S_M^{(\pm)})$ is a twistor spinor on M, then $\varphi|_F \in \Gamma(S_F)$ is a twistor on F.*

Proof. Let $\lambda \in C^\infty(F)$ be the function satisfying $\nabla_X^M \eta = \lambda X$. Then

$$\nabla_X^{S_F}(\varphi|_F) = \left(\nabla_X^{S_M^{(\pm)}} \varphi + \frac{1}{2}\kappa(\eta)\lambda X \cdot \eta \cdot \varphi \right)\Big|_F.$$

If $\varphi \in \Gamma(S_M^{(\pm)})$ is a twistor spinor, from Proposition 1, (4), it follows that

$$\kappa(X) X \cdot \nabla_X^{S_M^{(\pm)}} \varphi = \frac{1}{n+1} D_M^{(\pm)} \varphi$$

for each $X \in TF$ with $\kappa(X) = g(X, X) = \pm 1$. Hence,

$$\kappa(X) X \cdot \nabla_X^{S_F}(\varphi|_F) = \left(\frac{1}{n+1} D_M^{(\pm)} \varphi - \frac{1}{2}\kappa(\eta)\lambda\eta \cdot \varphi \right)\Big|_F.$$

The right-hand side is independent of $X \in TF$. Therefore, according to Proposition 1, $\varphi|_F$ is a twistor spinor on F. □

5. Twistor spinors on indecomposable, non-conformally flat Lorentzian symmetric spaces

Let us first consider the simply connected solvable Lorentzian symmetric space $M_{\underline{\lambda}}^n = (\mathbb{R}^n, g_{\underline{\lambda}})$, where

$$(g_{\underline{\lambda}})_{(s,t,x)} = 2\,\mathrm{d}s\,\mathrm{d}t + \sum_{j=1}^{n-2} \lambda_j x_j^2\,\mathrm{d}s^2 + \sum_{j=1}^{n-2} \mathrm{d}x_j^2,$$

and $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-2}), \lambda_i \in \mathbb{R}\backslash\{0\}, n \geq 3$. Let $\Lambda_0 := -\sum_{j=1}^{n-2} \lambda_j$. We fix the following global orthonormal basis on $M_{\underline{\lambda}}^n$:

$$\mathfrak{a}_{\bar{0}}(y) := \frac{\partial}{\partial s}(y) - \frac{1}{2}\left(\sum_{j=1}^{n-2}\lambda_j x_j^2 + 1\right)\frac{\partial}{\partial t}(y),$$

$$\mathfrak{a}_0(y) := \frac{\partial}{\partial s}(y) - \frac{1}{2}\left(\sum_{j=1}^{n-2}\lambda_j x_j^2 + 1\right)\frac{\partial}{\partial t}(y),$$

$$\mathfrak{a}_j(y) := \frac{\partial}{\partial x_j}(y), \quad j = 1, \ldots, n-2,$$

where $y = (s, t, x_1, \ldots, x_{n-2}) \in M^n_{\underline{\lambda}}$. The vector field $V(y) := (\partial/\partial t)(y)$ is isotropic and parallel. The Ricci tensor of $M^n_{\underline{\lambda}}$ is given by

$$\mathrm{Ric}(X) = \Lambda_0 \cdot g(X, V)V,$$

the scalar curvature R vanishes. Therefore, the Schouten tensor satisfies

$$K(X) = -\frac{1}{n-2}\Lambda_0 \cdot g(X, V)V. \tag{8}$$

For the Weyl tensor $W : \Lambda^2 M_{\lambda} \to \Lambda^2 M_{\underline{\lambda}}$ one has

$$\begin{aligned} W(\mathfrak{a}_{\bar{0}} \wedge \mathfrak{a}_j) &= W(\mathfrak{a}_0 \wedge \mathfrak{a}_j) \\ &= \big(\lambda_j + (1/(n-2))\Lambda_0\big)\, V \wedge \mathfrak{a}_j, \quad && j = 1, \ldots, n-2, \\ W(\mathfrak{a}_\alpha \wedge \mathfrak{a}_\beta) &= 0 && \text{for all other indices } \alpha, \beta, \end{aligned} \tag{9}$$

where $TM_{\underline{\lambda}}$ is identified with $T^*M_{\underline{\lambda}}$ using the metric $g_{\underline{\lambda}}$. In particular, $M_{\underline{\lambda}}$ is conformally flat if and only if $\underline{\lambda} = (\lambda, \ldots, \lambda)$, $\lambda \in \mathbb{R}\backslash\{0\}$.

Since $M_{\underline{\lambda}}$ is simply connected, it has an uniquely determined spin structure. We trivialize this spin structure using the global orthonormal basis $(\mathfrak{a}_{\bar{0}}, \mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{n-2})$ and identify the spinor fields with the smooth functions $C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$. The spinor derivative is defined by

$$\nabla^S_X \varphi = X(\varphi) + \frac{1}{2}\sum_{1\le k<l\le n} \varepsilon_k \varepsilon_l g(\nabla^{LC}_X s_k, s_l) s_k \cdot s_l \cdot \varphi,$$

where $(s_1, \ldots, s_n)$ is a local orthonormal basis and $\varepsilon_j = g(s_j, s_j) = \pm 1$. This gives for the spinor derivative on $M_{\underline{\lambda}}$

$$\nabla^S_{\partial/\partial t}\varphi = \frac{\partial}{\partial t}\varphi, \tag{10}$$

$$\nabla^S_{\partial/\partial s}\varphi = \frac{\partial}{\partial s}\varphi + \frac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j \cdot V \cdot \varphi, \tag{11}$$

$$\nabla^S_{\partial/\partial x_j}\varphi = \frac{\partial}{\partial x_j}\varphi, \tag{12}$$

$$\nabla^S_{\mathfrak{a}_{\bar{0}}}\varphi = \mathfrak{a}_{\bar{0}}(\varphi) + \frac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j \cdot V \cdot \varphi, \tag{13}$$

$$\nabla^S_{\mathfrak{a}_0}\varphi = \mathfrak{a}_0(\varphi) + \frac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j \cdot V \cdot \varphi. \tag{14}$$

The vector space $\Delta_{n,1}$ is isomorphic to $\Delta_{n-2,0} \otimes \mathbb{C}^2$. Let us denote by Δ_V the subspace

$$\Delta_V := \Delta_{n-2,0} \otimes \mathbb{C}u(-1) \subset \Delta_{n,1}.$$

Using formulas (1)–(3) one obtains that a spinor field $\varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$ satisfies $V \cdot \varphi = 0$ if and only if the image of φ lies in Δ_V.

Proposition 5. *The space $\mathcal{P}(M_{\underline{\lambda}})$ of parallel spinors of $M_{\underline{\lambda}}$ is*

$$\mathcal{P}(M_{\underline{\lambda}}) = \{\varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1}) | \varphi = \text{constant} \in \Delta_V\}.$$

In particular, $\dim \mathcal{P}(M_{\underline{\lambda}}) = \frac{1}{2} \cdot 2^{[n/2]}$.

Proof. From (8)–(10) it follows that $\varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$ is parallel if and only if φ depends only on s and

$$\frac{\partial\varphi}{\partial s} = -\frac{1}{2}\sum_{j=1}^{n-2}\lambda_j x_j \mathfrak{a}_j \cdot V \cdot \varphi. \tag{15}$$

Therefore,

$$0 = \frac{\partial^2\varphi}{\partial x_k \partial s} = -\frac{1}{2}\lambda_k \mathfrak{a}_k \cdot V \cdot \varphi.$$

Since $\lambda_k \neq 0$ and $\mathfrak{a}_k$ is space-like, this yields $V \cdot \varphi = 0$. Hence, because of (15), φ has to be constant. □

Proposition 6. *Let $M_{\underline{\lambda}}$ be non-conformally flat. Then each twistor spinor on $M_{\underline{\lambda}}$ is parallel. In particular*,

$$\dim \mathcal{T}(M_{\underline{\lambda}}) = \frac{1}{2} \cdot 2^{[n/2]}.$$

Proof. Let $\varphi \in C^\infty(M_{\underline{\lambda}}, \Delta_{n,1})$ be a twistor spinor. Then according to (7) of Proposition 2, $W(\eta) \cdot \varphi = 0$ for each 2-form η. Using (9) we obtain

$$\left(\lambda_j + \frac{1}{n-2}\Lambda_0\right)\mathfrak{a}_j \cdot V \cdot \varphi = 0, \quad j = 1, \dots, n-2.$$

Since $M_{\underline{\lambda}}$ is not conformally flat and $\mathfrak{a}_j$ is space-like, it follows that

$$0 = V \cdot \varphi. \tag{16}$$

Furthermore, we have

$$\nabla^S_X D\varphi \overset{(6)}{=} \frac{n}{2} K(X) \cdot \varphi.$$

Using (8) and (16) we obtain

$$\nabla^S_X D\varphi = -\frac{n}{2(n-2)}\Lambda_0 g(X,V)V\cdot\varphi = 0.$$

Hence, $D\varphi$ is parallel. From Proposition 5, it follows that $D\varphi$ is constant and $V\cdot D\varphi = 0$. Then the twistor equation and (10) yield

$$0 = \nabla^S_V\varphi + \frac{1}{n}V\cdot D\varphi = \frac{\partial}{\partial t}(\varphi).$$

Therefore, φ does not depend on t. The twistor equation implies that

$$-\mathfrak{a}_{\bar{0}}\cdot\nabla^S_{\mathfrak{a}_{\bar{0}}}\varphi = \mathfrak{a}_0\cdot\nabla^S_{\mathfrak{a}_0}\varphi = \frac{1}{n}D\varphi.$$

Then the formulas (13), (14) and (16) give

$$\begin{aligned} D\varphi &= -n\mathfrak{a}_{\bar{0}}\cdot\left(\frac{\partial}{\partial s} - \frac{1}{2}\left(\sum_{j=1}^{n-2}\lambda_j x_j^2 + 1\right)\frac{\partial}{\partial t}\right)\varphi \\ &= n\mathfrak{a}_0\cdot\left(\frac{\partial}{\partial s} - \frac{1}{2}\left(\sum_{j=1}^{n-2}\lambda_j x_j^2 - 1\right)\frac{\partial}{\partial t}\right)\varphi. \end{aligned}$$

Since φ does not depend on t we obtain

$$2D\varphi = n(\mathfrak{a}_0 - \mathfrak{a}_{\bar{0}})\cdot\frac{\partial}{\partial s}(\varphi) = nV\cdot\frac{\partial}{\partial s}(\varphi) = n\frac{\partial}{\partial s}(V\cdot\varphi)\overset{(16)}{=}0.$$

Therefore, φ is harmonic and the twistor equation implies that φ is parallel. □

Now, let (M^n, g) be a non-conformally flat, non-simply connected, indecomposable Lorentzian symmetric space of dimension $n \geq 3$. Then, according to Theorems 1, 2 and 4, (M^n, g) is isometric to $M^n_{\underline{\lambda}}/A$, where A is a discrete subgroup of the centralizer $Z_{\underline{\lambda}} := Z_{I(M_{\underline{\lambda}})}(G(M_{\underline{\lambda}}))$.

1. *Case*. There exist $i \in \{1,\dots,n-2\}$ such that $\lambda_i > 0$ or (i,j) such that $(\lambda_i/\lambda_j) \notin \mathbb{Q}^2$. Then $Z_{\underline{\lambda}} \simeq \mathbb{R} = \{\gamma_\alpha | \gamma_\alpha(s,t,x) = (s, t+\alpha, x), \alpha\in\mathbb{R}\}$.

   Let $\gamma \in A$. With respect to the global basic $(\mathfrak{a}_{\bar{0}}, \mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_{n-2})$ the differential $d\gamma_y$ corresponds to the matrix $\Gamma(y) \equiv E \in SO(n,1)$. Hence, $\Gamma^{\pm}(y) = \pm 1 \in \mathrm{Spin}(n,1)$. Therefore, we have two spin structures on $M = M_{\underline{\lambda}}/A$ corresponding to the homomorphisms $\mathrm{Hom}(A;\mathbb{Z}_2)$. If $\varepsilon \in \mathrm{Hom}(A,\mathbb{Z}_2)$ is not trivial, there are no ε-invariant constant spinor fields on $M_{\underline{\lambda}}$. From Propositions 3, 5 and 6 it follows that the twistor spinors on $M = M_{\underline{\lambda}}/A$ are given by

$$\mathcal{T}(M_{\underline{\lambda}}/A, \varepsilon) = \begin{cases} \{\varphi \in C^\infty(M, \Delta_V) | \varphi \text{ constant}\}, & \varepsilon \text{ trivial}, \\ \{0\}, & \varepsilon \text{ non-trivial}. \end{cases}$$

2. *Case*. Let $\lambda_j = -k_i^2 < 0$ and $(k_i/k_j) \in \mathbb{Q}$ for all $i, j = 1, \dots, n-2$. Then

$$Z_{\underline{\lambda}} \simeq \left\{ \gamma_{\underline{m},\alpha} | \gamma_{\underline{m},\alpha}(s,t,x) := (s+\beta, t+\alpha, (-1)^{m_1}x_1, \ldots, (-1)^{m_{n-2}}x_{n-2}), \right.$$
$$\text{where } \alpha \in \mathbb{R}, \quad \underline{m} = (m_1, \ldots, m_{n-2}) \in \mathbb{Z}^{n-2}, \beta = \pi \cdot \frac{m_i}{k_i},$$
$$\left. i = 1, \ldots, n-2 \right\} \simeq \mathbb{Z} \oplus \mathbb{R}.$$

A discrete subgroup $A_{\underline{m},\alpha} \subset Z_{\underline{\lambda}}$ is generated by $\gamma_{\underline{m},0}$ and $\gamma_{0,\alpha}$. Let us suppose that $\sum_{i=1}^{n-2} m_i$ is even since otherwise $M_{\underline{\lambda}}/A_{\underline{m},\alpha}$ is not orientable. $(\mathrm{d}\gamma_{\underline{m},\alpha})_y$ corresponds to the matrix

$$\Gamma(y) = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & (-1)^{m_1} & & \\ & & & \ddots & \\ 0 & & & & (-1)^{m_{n-2}} \end{pmatrix}.$$

Hence $\tilde{\Gamma}^{\pm}(y) = \pm e_1^{m_1} \cdots e_{n-2}^{m_{n-2}}$. Let $m_{i_1}, \ldots, m_{i_s}$ be the odd elements in the tupel $\underline{m}$ ($s \in 2\mathbb{Z}$), and let us denote by $\omega_{\underline{m}} \in \mathrm{Spin}(n,1)$ the element

$$\omega_{\underline{m}} = e_{i_1} \cdots e_{i_s}.$$

Then because of $\omega_{\underline{m}}^2 = (-1)^{s/2}$, $\omega_{\underline{m}}$ is an involution on $\Delta_{n-2,0}$ if $s \equiv 0(4)$ and an almost complex structure if $s \equiv 2(4)$. The eigenspaces of $\omega_{\underline{m}}$ to the eigenvalues ± 1 and $\pm \mathrm{i}$, respectively, have the same dimension (see formulas (1) and (2)).

The manifold $M_{\underline{\lambda}}/A_{\underline{0},\alpha}$, $\alpha \neq 0$, has two spin structures and the twistor spinors are given as in Case 1. $M_{\underline{\lambda}}/A_{\underline{m},0}$, $\underline{m} \neq \underline{0}$, has two spin structures, described by the homomorphisms $\varepsilon_{\pm} \in \mathrm{Hom}(A_{\underline{m},0}, \mathrm{Spin}(n,1))$ given by $\varepsilon_{\pm}(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}$. Then, according to Propositions 3, 5 and 6 the twistor spinors on $M = M_{\underline{\lambda}}/A_{\underline{m},0}$ are

$$\mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}, \varepsilon_{\pm}) = \left\{ \varphi \in C^{\infty}(M_{\underline{\lambda}}, \Delta_{n,1}) | \varphi(y) \equiv v \otimes u(-1), \right.$$
$$\left. \text{where } v \in \Delta_{n-2,0} \text{ and } \omega_{\underline{m}} \cdot v = \pm v \right\}.$$

Hence,

$$\dim \mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}, \varepsilon_{\pm}) = \begin{cases} 0 & \text{if } s \equiv 2(4), \\ \frac{1}{4} \cdot 2^{[n/2]} & \text{if } s \equiv 0(4). \end{cases}$$

The manifold $M_{\underline{\lambda}}/A_{\underline{m},\alpha}$, $\underline{m} \neq 0, \alpha \neq 0$, has four spin structures corresponding to the homomorphisms $\varepsilon \in \mathrm{Hom}(A_{\underline{m},\alpha}; \mathrm{Spin}(n,1))$ given by $\varepsilon(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}, \varepsilon(\gamma_{0,\alpha}) = \pm 1$. Hence,

$$\mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}, \varepsilon) = \begin{cases} \{0\} & \varepsilon(\gamma_{0,\alpha}) = -1, \\ \mathcal{T}(M_{\underline{\lambda}}/A_{\underline{m},0}, \varepsilon_{\pm}), & \varepsilon(\gamma_{0,\alpha}) = 1,\ \varepsilon(\gamma_{\underline{m},0}) = \pm\omega_{\underline{m}}. \end{cases}$$

Summing up, we have the following proposition.

Proposition 7. *Let (M^n, g) be an indecomposable, non-conformally flat Lorentzian symmetric spin manifold of dimension $n \geq 3$. Then each twistor spinor is parallel and the dimension of the space of twistor spinors is*

$$\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]},$$

where $q = 0, \frac{1}{4}$ *or* $\frac{1}{2}$, *depending on the fundamental group* $\pi_1(M)$ *and on the spin structure*.

6. Twistor spinors on indecomposable conformally flat Lorentzian symmetric spaces of non-constant sectional curvature

According to Theorems 1, 2 and 4 there are two isometry classes of indecomposable, conformally flat, simply connected Lorentzian symmetric spaces of dimension $n \geq 3$ and non-constant sectional curvature, namely

$$M^n_\pm := (\mathbb{R}^n, g_\pm), \qquad (g_\pm)_{(s,t,x)} = 2\,\mathrm{d}s\,\mathrm{d}t \pm \|x\|^2\,\mathrm{d}s^2 + \sum_{j=1}^{n-2} \mathrm{d}x_j^2.$$

We known that $\dim \mathcal{T}(M^n_\pm) = 2 \cdot 2^{[n/2]}$. The twistor spinors are given by the following formula. Let $\omega_1, \omega_2, \omega_3, \omega_4 \in \Delta_{n-2,0}$ and let us denote by $\varphi_{\omega_1,\omega_2,\omega_3,\omega_4} \in C^\infty(M^n_\pm, \Delta_{n,1})$ the following smooth functions

$$\begin{aligned}\varphi_{\omega_1,\omega_2,\omega_3,\omega_4}(s,t,x) := {} & (\mp f'(s) w_3 - f(s) w_4 + x \cdot w_1) \otimes u(1) + [-2\omega_1 t + \omega_2 \\ & + x \cdot (f(s)\omega_3 + f'(s)\omega_4)] \otimes u(-1),\end{aligned}$$

$$f(s) = \begin{cases} \sinh(s) & \text{for } M^n_+, \\ \sin(s) & \text{for } M^n_-. \end{cases}$$

Proposition 8. *The twistor spinors on* $M^n_\pm$ *are*

$$\mathcal{T}(M^n_\pm) = \{\varphi_{w_1,w_2,w_3,w_4} | w_1, w_2, w_3, w_4 \in \Delta_{n-2,0}\}.$$

Proof. We use the identification

$$\begin{aligned}&\Delta_{n,1} \simeq \Delta_{n-2,0} \otimes \Delta_{2,1} \xrightarrow{\sim} \Delta_{n-2,0} \oplus \Delta_{n-2,0}, \\ &\quad \varphi = \varphi_1 \otimes u(1) + \varphi_2 \otimes u(-1) \mapsto (\varphi_1, \varphi_2).\end{aligned}$$

Then according to (1) and (2) the Clifford multiplication corresponds to

$$\begin{aligned}&X \cdot \varphi = (-X \cdot \varphi, X \cdot \varphi_2) \text{ if } X \in \mathrm{span}(\mathfrak{a}_1, \ldots, \mathfrak{a}_{n-2}), \qquad \mathfrak{a}_{\bar{0}} \cdot \varphi = (-\varphi_2, -\varphi_1), \\ &\mathfrak{a}_0 \cdot \varphi = (-\varphi_2, \varphi_1), \qquad V \cdot \varphi = (0, 2\varphi_1).\end{aligned}$$

For the spinor derivative we obtain

$$\begin{aligned}&\nabla^S_{\mathfrak{a}_{\bar{0}}} \varphi = (\mathfrak{a}_{\bar{0}}(\varphi_1), \mathfrak{a}_{\bar{0}}(\varphi_2) \pm x \cdot \varphi_1), \qquad \Delta^S_{\mathfrak{a}_0} \varphi = (\mathfrak{a}_0(\varphi_1), \mathfrak{a}_0(\varphi_2)) \pm x \cdot \varphi_1), \\ &\nabla^S_{\mathfrak{a}_k} \varphi = (\mathfrak{a}_k(\varphi_1), \mathfrak{a}_k(\varphi_2)), \quad k = 1, \ldots, n-2.\end{aligned}$$

Let $\varphi = (\varphi_1, \varphi_2)$ be a twistor spinor on $M^n_\pm$. Then, according to Proposition 1, there exists a spinor field $\psi = (\psi_1, \psi_2)$ on $M^n_\pm$ such that

$$(\psi_1, \psi_2) = \mathfrak{a}_k \cdot \nabla^S_{\mathfrak{a}_k} \varphi = (-\mathfrak{a}_k \cdot \mathfrak{a}_k(\varphi_1), \mathfrak{a}_k \cdot \mathfrak{a}_k(\varphi_2)) \tag{17}$$

for each $k = 1, 2, \ldots, n-2$. Therefore, $\varphi_1(s, t, \cdot)$ and $\varphi_2(s, t, \cdot)$ are twistor spinors on the Euclidean space $\mathbb{R}^{n-2}$. Hence,

$$\varphi_i(s, t, x) = u_i(s, t) + x \cdot v_i(s, t), \quad i = 1, 2,$$

where $u_i, v_i : \mathbb{R}^2 \to \Delta_{n-2,0}$. From (17) follows

$$\psi_1(s, t, x) = v_1(s, t) \quad \text{and} \quad \psi_2(s, t, x) = -v_2(s, t).$$

Furthermore, $\psi = (\psi_1, \psi_2)$ satisfies

$$(\psi_1, \psi_2) = -\mathfrak{a}_{\bar{0}} \cdot \nabla^S_{\mathfrak{a}_{\bar{0}}} \varphi = \mathfrak{a}_0 \cdot \nabla^S_{\mathfrak{a}_0} \varphi.$$

Therefore

$$v_1 = \left(\frac{\partial}{\partial s} - \frac{1}{2}(\pm\|x\|^2 + 1)\frac{\partial}{\partial t} \right)(u_2 + x \cdot v_2) \pm x \cdot (u_1 + x \cdot v_1), \tag{18}$$

$$v_1 = -\left(\frac{\partial}{\partial s} - \frac{1}{2}(\pm\|x\|^2 - 1)\frac{\partial}{\partial t} \right)(u_2 + x \cdot v_2) \mp x \cdot (u_1 + x \cdot v_1), \tag{19}$$

$$-v_2 = \left(\frac{\partial}{\partial s} - \frac{1}{2}(\pm\|x\|^2 + 1)\frac{\partial}{\partial t} \right)(u_1 + x \cdot v_1), \tag{20}$$

$$-v_2 = \left(\frac{\partial}{\partial s} - \frac{1}{2}(\pm\|x\|^2 - 1)\frac{\partial}{\partial t} \right)(u_1 + x \cdot v_1), \tag{21}$$

Adding Eqs. (18) and (19) gives $2v_1 = -(\partial/\partial t)u_2 - x \cdot (\partial/\partial t)v_2$ and after differentiation with respect to x_k, $0 = -\mathfrak{a}_k(\partial/\partial t)v_2$. Hence,

$$\frac{\partial}{\partial t}v_2 = 0 \quad \text{and} \quad v_1 = -\frac{1}{2}\frac{\partial}{\partial t}u_2. \tag{22}$$

Using this and subtracting Eq. (19) from Eq. (18), we obtain

$$0 = \mp x \cdot u_1 - \frac{\partial}{\partial s}u_2 - x \cdot \frac{\partial}{\partial s}v_2. \tag{23}$$

Differentiating Eq. (23) shows that

$$u_1 = \mp\frac{\partial}{\partial s}v_2 \quad \text{and} \quad \frac{\partial}{\partial t}u_1 = 0. \tag{24}$$

Inserting this in (23) and using (22) we obtain

$$\frac{\partial}{\partial s}u_2 = 0 \quad \text{and} \quad \frac{\partial}{\partial s}v_1 = 0. \tag{25}$$

Hence, $u_2 = u_2(t)$, $v_1 = v_1(t)$, $v_2 = v_2(s)$ and $u_1 = u_1(s)$. By subtracting Eq. (20) from Eq. (21), we obtain

$$0 = x \cdot \frac{\partial}{\partial t}v_1 + \frac{\partial}{\partial t}u_1 = x \cdot v_1'(t).$$

Therefore, we have $v_1(t) \equiv w_1 \in \Delta_{n-2}$ and, because of (22), $u_2(t) = -2tw_1 + w_2$. Adding Eq. (21) with Eq. (20) yields

$$-2v_2 = 2u_1'(s) \mp \|x\|^2 v_1'(t) = 2u_1'(s),$$

so that, regarding (24), $v_2(s) = \pm v_2''(s)$. Therefore,

$$v_2(s) = f(s)w_3 + f'(s)w_4 \quad \text{and} \quad u_1(s) = \mp f'(s)w_3 - f(s)w_4,$$

$$f(s) = \begin{cases} \sinh(s) & \text{for } M_+^n, \\ \sin(s) & \text{for } M_-^n. \end{cases}$$

Consequently, the twistor spinor φ is of the form $\varphi = \varphi_{w_1,w_2,w_3,w_4}$. □

Now let (M^n, g) be an indecomposable, conformally flat non-simply connected Lorentzian symmetric space of dimension $n \geq 3$ and non-constant sectional curvature. Then (M^n, g) is isometric to M_+^n/A or to M_-^n/A, where A is a discrete subgroup of

$$Z_+ := Z_{I(M_+)}(G(M_+)) = \{\varphi_\alpha | \varphi_\alpha(s,t,x) = (s, t+\alpha, x);\ \alpha \in \mathbb{R}\}$$

in the first and of

$$\begin{aligned} Z_- := Z_{I(M_-)}(G(M_-)) &= \{\varphi_{m,\alpha} | \varphi_{m,\alpha}(s,t,x) \\ &= (s+m\pi, t+\alpha, (-1)^m x);\ m \in \mathbb{Z}, \alpha \in \mathbb{R}\} \end{aligned}$$

in the second Case.

1. *Case*. $M = M_+^n/A_\alpha$, $A_\alpha = \mathbb{Z}\varphi_\alpha$. Then there are two spin structures corresponding to $\varepsilon \in \mathrm{Hom}(A_\alpha, \mathbb{Z}_2)$. Propositions 3 and 8 show

$$\mathcal{T}(M,\varepsilon) = \begin{cases} \{\varphi_{0,w_2,w_3,w_4} | w_2, w_3, w_4 \in \Delta_{n-2,0}\}, & \varepsilon = 1, \\ \{0\}, & \varepsilon \neq 1. \end{cases}$$

2. *Case*. $M = M_-^n/A_{m,\alpha}$, $A_{m,\alpha} = \langle \varphi_{m,0}, \varphi_{0,\alpha} \rangle$. If m is even and $\alpha \neq 0$, we have the same result as in Case 1, since $f(s) = \sin(s)$ is $2\pi\mathbb{Z}$-invariant. If m is odd, M is orientable only if n is even. Then M^{2k} has two spin structures if $\alpha = 0$ and four spin structures if $\alpha \neq 0$. Propositions 3 and 8 show

$$\mathcal{T}(M_-^{2k}/A_{m,0,\varepsilon}) = \begin{cases} \{0\}, & \text{if } n = 2k \equiv 0(4), \\ \{\varphi_{w_1,w_2,w_3,w_4} | w_1, & \\ w_2 \in \Delta_{n-2,0}^{\pm}, w_3, w_4 \in \Delta_{n-2,0}^{\mp}\}, & \text{if } n = 2k \equiv 2(4), \\ & \varepsilon(\varphi_{m,0}) = \pm e_1 \cdots e_{n-2}, \end{cases}$$

$$\mathcal{T}(M_-^{2k}/A_{m,\alpha,\varepsilon}) = \begin{cases} \{0\}, & \text{if } \varepsilon(\varphi_{0,\alpha}) = -1 \text{ or} \\ & n = 2k \equiv 0(4), \\ \{\varphi_{0,w_2,w_3,w_4} | w_2 \in \Delta_{n-2,0}^{\pm}, & \\ w_3, w_4 \in \Delta_{n-2,0}^{\mp}\}, & \text{if } n = 2k \equiv 2(4), \\ & \varepsilon(\varphi_{0,\alpha}) = 1 \text{ and } \varepsilon(\varphi_{m,0}) \\ & = \pm e_1 \cdots e_{n-2}. \end{cases}$$

Summing up, we have in particular the following proposition.

Proposition 9. *Let (M^n, g) be an indecomposable, conformally flat Lorentzian spin manifold (M^n, g) of non-constant sectional curvature and dimension $n \geq 3$. Then the dimension of the space of twistor spinors is*

$$\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]},$$

where $q = 0, \frac{3}{4}, 1, \frac{3}{2}$ or 2 depending on the fundamental group $\pi_1(M)$ and on the spin structure.

7. Twistor spinors on Lorentzian symmetric spaces of constant sectional curvature

Let $\psi_{u,v} \in C^\infty(\mathbb{R}^{n+1,k}, \Delta_{n+1,k})$ denote the twistor spinors on the pseudo-Euclidean space $\mathbb{R}^{n+1,k}$:

$$\psi_{u,v}(x) := u + x \cdot v, \quad u, v \in \Delta_{n+1,k}.$$

The pseudo-sphere $S_1^n(r) \subset \mathbb{R}^{n+1,1}$ and the pseudo-hyperbolic space $H_1^n(r)$ are umbilic hypersurfaces. Using the identification of the spinor bundle of the hypersurface with that of the external space (see Section 4) we obtain from Proposition 4 the following.

Proposition 10. *The twistor spinors on $S_1^n(r)$ and $H_1^n(r)$ with the induced spin structure are*

$$\mathcal{T}(S_1^n(r)) = \left\{ \psi_{u,v|S_1^n(r)} \;\middle|\; \begin{array}{ll} u, v \in \Delta_{n+1,1} & \text{if } n \equiv 0(2) \\ u \in \Delta^+_{n+1,1}, v \in \Delta^-_{n+1,1} & \text{if } n \equiv 1(2) \end{array} \right\},$$

$$\mathcal{T}(H_1^n(r)) = \left\{ \psi_{u,v|H_1^n(r)} \;\middle|\; \begin{array}{ll} u, v \in \Delta_{n+1,2} & \text{if } n \equiv 0(2) \\ u \in \Delta^+_{n+1,2}, v \in \Delta^-_{n+1,2} & \text{if } n \equiv 1(2) \end{array} \right\}.$$

The Lorentzian manifold $S_1^n(r)/_{\{\pm I\}}$ is orientable if and only if n is odd, hence let n be odd. The volume form $w_{n+1,1} = e_1 \cdots e_{n+1} \in \mathrm{Spin}(n+1, 1)$ satisfies $w^2_{n+1,1} = (-1)^{(((n+1)/2)+1)}$. Therefore, $S_1^n(r)/_{\{\pm I\}}$ has no spin structure, if $n \equiv 3(4)$ and two spin structures, if $n \equiv 1(4)$. The spinor fields to these different spin structures can be identified with the invariant functions $C^\infty(S_1^n(r), \Delta^+_{n+1,1})^{\varepsilon\pm}$, where $\varepsilon_\pm$ is the $\mathbb{Z}_2$-action given by

$$(\varepsilon_\pm(-1)\varphi)(x) = \pm w_{n+1,1} \cdot \varphi(-x) = \pm\varphi(-x).$$

From Propositions 3 and 10, it follows that

$$\mathcal{T}(S_1^{4k+1}(r)/_{\{\pm I\}}, \varepsilon) = \begin{cases} \{\psi_{u^+,0|S_1^{4k+1}} | u^+ \in \Delta^+_{4k+2,1}\} & \text{if } \varepsilon = \varepsilon_+, \\ \{\psi_{0,v^-|S_1^{4k+1}} | v^- \in \Delta^-_{4k+2,1}\} & \text{if } \varepsilon = \varepsilon_-. \end{cases}$$

Now, let us consider a Lorentzian symmetric space M^n of constant negative sectional curvature. Then M^n is isometric to a Lorentzian covering of $H_1^n(r)/_{\{\pm I\}}$. Let

$$\tilde{\pi} : \widetilde{H_1^n(r)} = \mathbb{R} \times \mathbb{R}^{n-1} \to H_1^n(r) \subset \mathbb{R}^{2,2} \times \mathbb{R}^{n-2},$$
$$(t, x) \mapsto \left(\sqrt{r^2 + \|x\|^2} \cos t, \sqrt{r^2 + \|x\|^2} \sin t, x \right)$$

be the universal Lorentzian covering of $H_1^n(r)$. Let $\hat{Q}$ denote the reduction of the trivial spin structure Q of $\mathbb{R}^{n+1,2}$ to the subgroup $\mathrm{Spin}(n,1)$ given by the Gauss map. Then $\tilde{Q} := \tilde{\pi}^*\hat{Q}$ is the uniquely determined spin structure of $\widetilde{H_1^n}(r)$. The spinor fields of $\widetilde{H_1^n}(r)$ can be identified with the smooth functions $C^\infty(\widetilde{H_1^n}(r), \Delta^{(\pm)}_{n+1,2})$, the twistor spinors are given by

$$\mathcal{T}(\widetilde{H_1^n}(r)) = \left\{ \tilde{\psi}_{u,v} := \psi_{u,v|H_1^n(r)} \circ \tilde{\pi} \;\middle|\; \psi_{u,v|H_1^n(r)} \in \mathcal{T}(H_1^n(r)) \right\}.$$

Let

$$\pi_m : N_m^n \to H_1^n(r), \qquad (\sqrt{\cdot}\cos t, \sqrt{\cdot}\sin t, x) \mapsto (\sqrt{\cdot}\cos(mt), \sqrt{\cdot}\sin(mt), x)$$

($\sqrt{\cdot} = \sqrt{r^2 + \|x\|^2}$), be the Lorentzian covering of $H_1^n(r)$ with respect to $m\mathbb{Z} \subset \pi_1(H_1^n(r)) = \mathbb{Z}, m = 1, 2, 3, \ldots$. The manifold N_m^n has two spin structures. The corresponding spinor fields are given by the $\varepsilon_m^\pm$-invariant functions $C^\infty(\widetilde{H_1^n(r)}, \Delta^{(\pm)}_{n+1,2})^{\varepsilon_m^\pm}$, where $\varepsilon_m^\pm$ is the $m\mathbb{Z}$-action:

$$(\varepsilon_m^\pm(mz)\varphi)(t,x) = (\pm 1)^z \varphi(t + 2\pi m z, x).$$

Therefore, the twistor spinors on N_m^n are

$$\mathcal{T}(N_m^n, \varepsilon) = \begin{cases} \left\{ \{\psi_{u,v|H_1^n(r)} \circ \pi_m \;\middle|\; \psi_{u,v|H_1^n(r)} \in \mathcal{T}(H_1^n(r)) \right\} & \text{if } \varepsilon = \varepsilon_m^+, \\ \{0\} & \text{if } \varepsilon = \varepsilon_m^-. \end{cases}$$

Finally, let us consider the manifolds $N_m^n/_{\{\pm I\}}$. Since $N_m^n/_{\{\pm I\}}$ is orientable if and only if n is odd, let n be odd. For the volume form $w_{n+1,2} = e_1 \cdots e_{n+1} \in \mathrm{Spin}(n+1,2)$ we have $w_{n+1,2}^2 = (-1)^{(((n+1)/2)+2)}$. Therefore, there is no spin structure on $N_m^n/_{\{\pm I\}}$ if $n \equiv 1(4)$ and there are four spin structures in case $n \equiv 3(4)$. The spinor fields are given by the functions $C^\infty(\widetilde{H_1^n(r)}, \Delta^{(+)}_{n+1,2})^{(\varepsilon_m^\pm, \delta^\pm)}$, invariant under the $m\mathbb{Z}$-action $\varepsilon_m^\pm$ and the $\mathbb{Z}_2$-action $\delta^\pm$ given by

$$(\delta^\pm(-1)\varphi)(t,x) = \pm w_{n+1,2} \cdot \varphi(t + m\pi, -x) = \pm\varphi(t + m\pi, -x).$$

Then, the twistor spinors are

$$\mathcal{T}(N_m^{4k+3}/_{\{\pm I\}}, \varepsilon) = \begin{cases} \{0\}, & \varepsilon = (\varepsilon_m^-, \delta^\pm) \quad \text{or} \\ & \varepsilon = (\varepsilon_m^+, \delta^-), \quad m \equiv 0(2), \\ \{\tilde{\psi}_{u^+,0} \;|\; u^+ \in \Delta^+_{4k+4,2}\}, & \varepsilon = (\varepsilon_m^+, \delta^+), \\ \{\tilde{\psi}_{0,v^-} \;|\; v^- \in \Delta^+_{4k+4,2}\}, & \varepsilon = (\varepsilon_m^+, \delta^-), m \equiv 1(2). \end{cases}$$

Summing up, we have in particular the following theorem.

Proposition 11. *Let (M^n, g) be a Lorentzian symmetric spin manifold of constant sectional curvature $k \neq 0$ and dimension $n \geq 3$, then the dimension of the space of twistor spinor is*

$$\dim \mathcal{T}(M^n, g) = q \cdot 2^{[n/2]},$$

where $q = 0, 1,$ or 2 depending on $\pi_1(M)$ and on the spin structure.

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References

[1] D.V. Alekseevsky, V. Cortes, C. Devchand, U. Semmelmann, Killing spinors are Killing vector fields in Riemannian supergeometry, J. Geom. Phys. 26 (1–2) (1998) 51–78.
[2] H. Baum, Lorentzian twistor spinors and CR-geometry, Diff. Geom. Appl. 11 (1) (1999) 69–96.
[3] H. Baum, T. Friedrich, R. Grunewald, I. Kath, Twistors and Killing spinors on Riemannian manifolds, Teubner-Texte zur Mathematik, Vol. 124, Teubner, Stuttgart, 1991.
[4] Ch. Bohle, Killing spinors on Lorentzian manifolds, SFB 288 preprint, 1999, No. 417.
[5] M. Cahen, Y. Kerbrat, Champs de vecteurs conformes et transformations conformes des spaces Lorentzian symetriques, J. Math. Pures Appl. 57 (1978) 99–132.
[6] M. Cahen, J. Leroy, M. Parker, F. Tricerri, L. Vanhecke, Lorentzian manifolds modelled on a Lorentz symmetric space, J. Geom. Phys. 7 (4) (1990) 571–581.
[7] M. Cahen, M. Parker, Pseudo-Riemannian symmetric spaces, Mem. Am. Math. Soc. 24 (229) (1980) 1–108.
[8] M. Cahen, N. Wallach, Lorentzian symmetric spaces, Bull. Am. Math. Soc. 76 (3) (1970) 585–591.
[9] I. Kath, Killing Spinors on Pseudo-Riemannian Manifolds, Habilitationsschrift Humboldt-Universität Berlin, Berlin, 1999.
[10] W. Kühnel, H.-B. Rademacher, Asymptotically Euclidean manifolds and twistor spinors, Commun. Math. Phys. 196 (1998) 67–76 .
[11] J. Lewandowski, Twistor equation in a curved space–time, Classical Quantum Gravity 8 (1991) 11–17.
[12] P. Nieuwenhuizen, N.P. Warner, Integrability conditions for Killing spinors, Commun. Math. Phys. 93 (1984) 277–284.
[13] R. Penrose, Twistor algebra, J. Math. Phys. 8 (1967) 345–366.
[14] R. Penrose, W. Rindler, Spinors and Space–time II, Cambridge University Press, Cambridge, 1986.
[15] J.A. Wolf, Spaces of Constant Curvature, Publish or Perish, Berkeley, CA, 1984.